Top-k Selection based on Adaptive Sampling of Noisy Preferences

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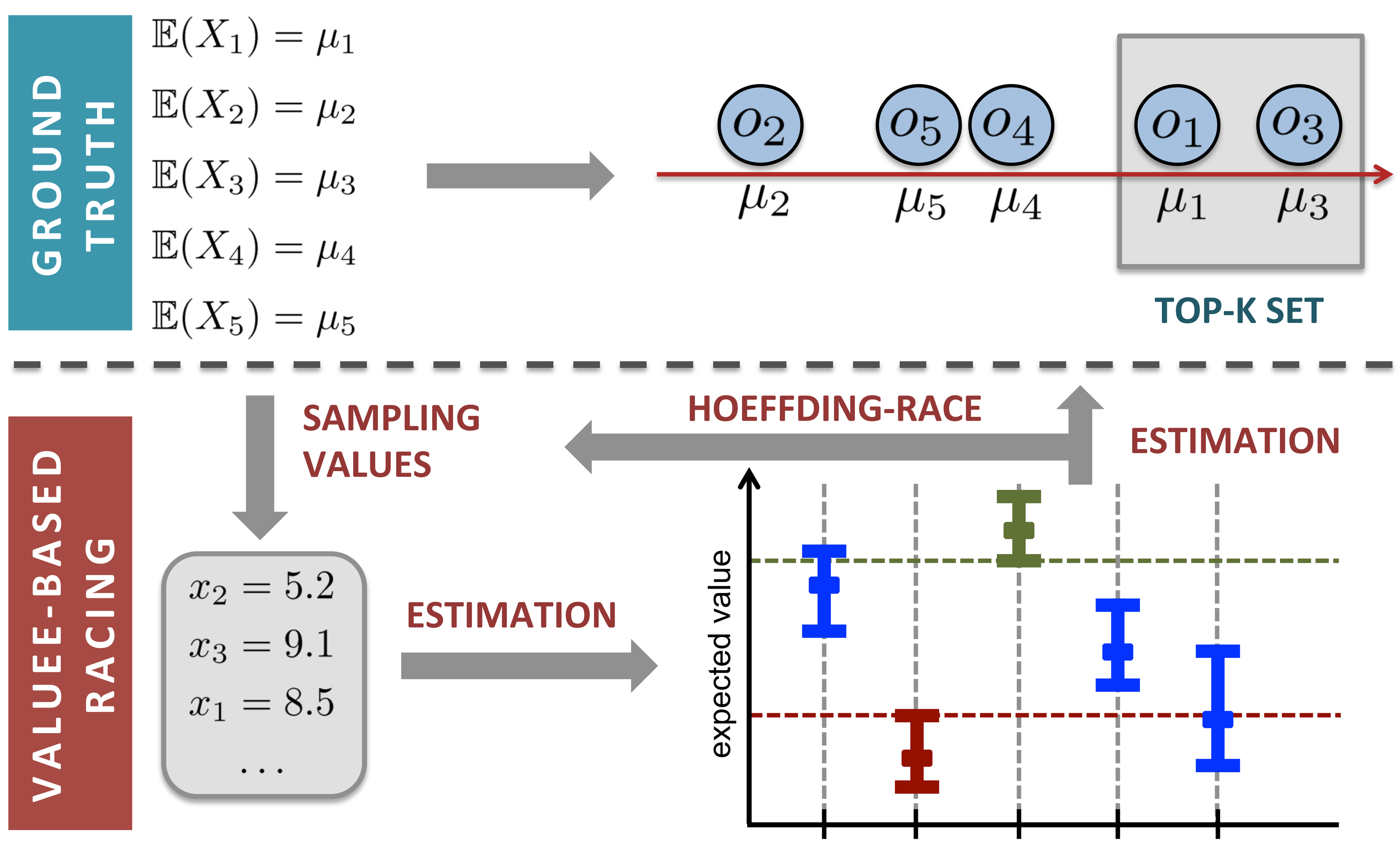
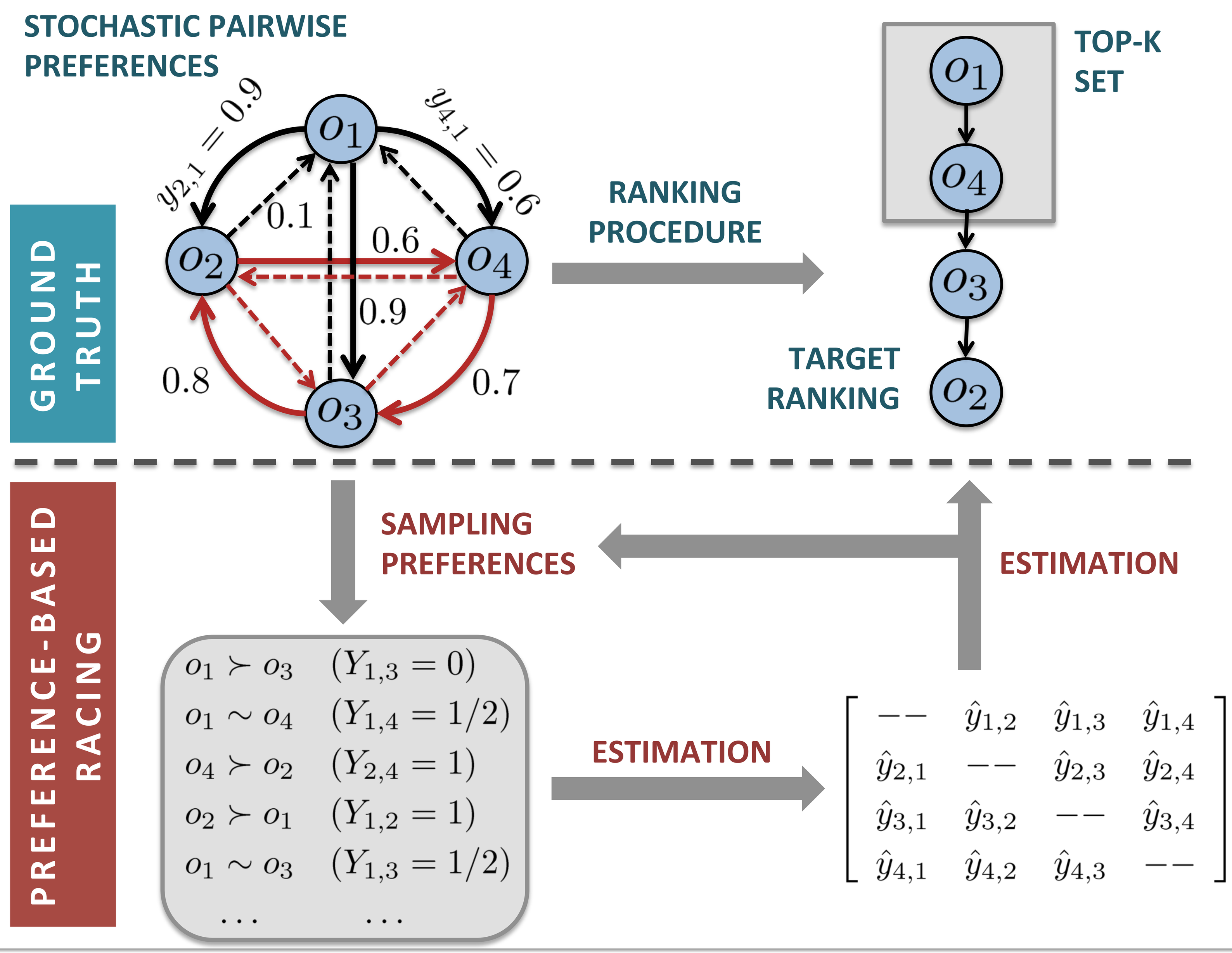
Value-based Top-k Selection (TKS) [1]



Preference-based Top-k Selection (TKS)



Resolving Preferential Cycles

$y_{i,j} = \mathbb{E}[Y_{i,j}]$, $\mathbf{Y} = [y_{i,j}]_{K \times K}$

1. Copeland's ranking: (CO): $o_i \prec^{\mathrm{CO}} o_j$ if and only if $d_i < d_j$, where
$$d_i = \#\{k \in [K] \,|\, 1/2 < y_{i,k}\} \ ,$$
   - An option o_i is preferred to o_j whenever o_i "beats" more options than o_j does.
2. Sum of expectations (SE) ranking: CO: $o_i \prec^{\mathrm{SE}} o_j$ if and only if
$$y_i = \frac{1}{K-1}\sum_{k \neq i} y_{i,k} < \frac{1}{K-1}\sum_{k \neq j} y_{j,k} = y_j \ .$$
3. The idea of the Random walk (RW) ranking is to handle the matrix $\mathbf{Y}$ as a transition matrix $\mathbf{S}$ of a Markov chain and order the options based on its stationary distribution.

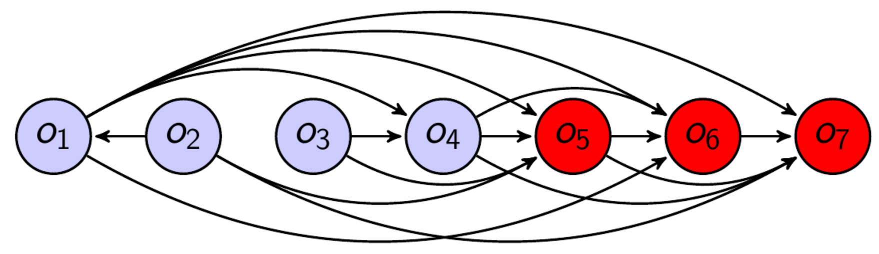
Theorem (Expected sample complexity for Copeland's ranking)

Let $\mathcal{O} = \{o_1, \dots, o_K\}$ be a set of options such that $\Delta_{i,j} = y_{i,j} - 1/2 \neq 0$ for all $i, j \in [K]$. The expected number of pairwise comparison taken by $\mathrm{PBR\text{-}CO}$ is bounded by
$$\sum_{i=1}^{K}\sum_{j \neq i}\left\lceil \frac{1}{2\Delta_{i,j}^2}\log\frac{2K^2 n_{\max}}{\delta}\right\rceil \ .$$
Moreover, the probability that no optimal solution is found by $\mathrm{PBR\text{-}CO}$ is at most δ if $n_{i,j} \leq n_{\max}$ for all $i, j \in [K]$.

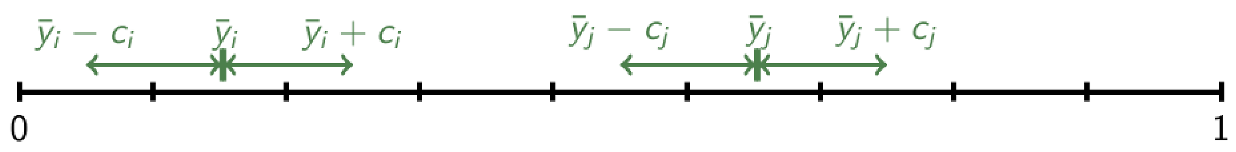
Sampling Strategies

Sampling strategies

- **Algorithm** $\mathrm{PBR}\,(Y_{1,1}, \dots, Y_{K,K}, \kappa, n_{\max}, \delta)$
  1. Initially, sample each $Y_{i,j}$
  2. In each iteration, calculate $\bar{y}_{i,j} = \frac{1}{n_{i,j}}\sum_{\ell=1}^{n_{i,j}} y_{i,j}^{\ell}$ and its confidence interval $[\bar{y}_{i,j} - c_{i,j}, \bar{y}_{i,j} + c_{i,j}]$ with $c_{i,j} = \sqrt{\frac{1}{2n_{i,j}}\log\frac{2K^2 n_{\max}}{\delta}}$
  3. and decide which $Y_{i,j}$ will be sampled next based on the specific ranking procedure of interest
- Copeland's ranking: $1/2 \notin [\bar{y}_{i,j} - c_{i,j}, \bar{y}_{i,j} + c_{i,j}]$



- Sum of expectations (SE): $y_i \in [\bar{y}_i - c_i, \bar{y}_i + c_i]$ where $\bar{y}_i = \frac{1}{K-1}\sum_{k \neq i}\bar{y}_{i,k}$ and $c_i = \frac{1}{K-1}\sum_{j \neq i} c_{i,j}$

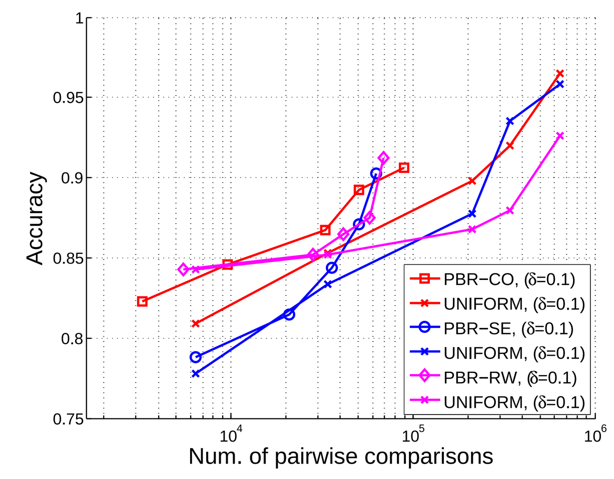


- Random walk (RW) ranking: transform $\bar{\mathbf{Y}}$ into *stochastic matrix* $\bar{\mathbf{S}}$
  - $s_{i,j} \in [\bar{s}_{i,j} - \mathfrak{c}_{i,j}, \bar{s}_{i,j} + \mathfrak{c}_{i,j}]$, where $\mathfrak{c}_{i,j} = \frac{K}{3}\max_k c_{i,k}\sum_{\ell}\bar{y}_{\ell,i}$ (see Lemma 1-2 in [2])
  - Let $\mathbf{S}\mathrm{v} = 1\mathrm{v}$ and $\bar{\mathbf{S}}\bar{\mathrm{v}} = 1\bar{\mathrm{v}}$. Then, according to [3], we have
$$\|\mathrm{v} - \bar{\mathrm{v}}\|_{\max} \leq \text{const.} \times \max_{1 \leq i \leq K}\sum_j |s_{i,j} - \bar{s}_{i,j}| \leq \text{const.} \times \max_{1 \leq i \leq K}\sum_j \mathfrak{c}_{i,j}$$

Experiments: Bundesliga

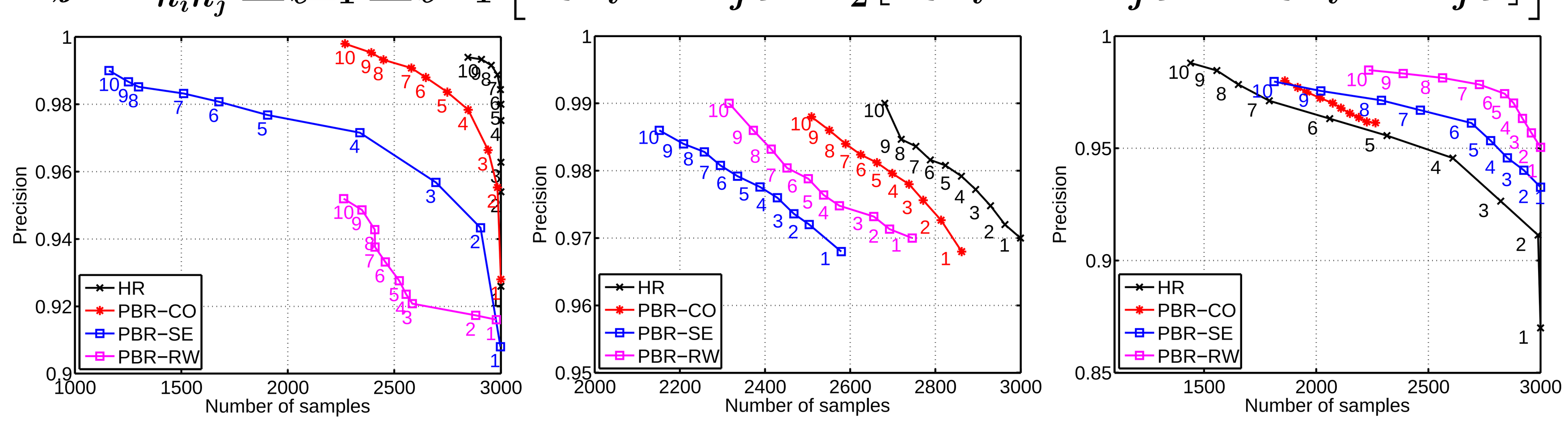
- soccer matches of the last ten seasons from the German Bundesliga
- uniform sampling as baseline
- $\delta = 0.1, \kappa = 3, n_{\max} = \{100, 500, 1000, 5000, 10000\}$

Team	W	L	T	$\prec^{\mathrm{CO}}$	$\prec^{\mathrm{SE}}$	$\prec^{\mathrm{RW}}$
B. München	77	33	30	*1	*1	*1
B. Dortmund	56	49	35	*3	*2	5
B. Leverkusen	55	49	36	5	4	*2
VfB Stuttgart	55	53	32	*2	5	4
Schalke 04	54	47	39	4	*3	*3
W. Bremen	52	51	37	6	6	6
VfL Wolfsburg	44	66	30	7	7	7
Hannover 96	30	75	35	8	8	8



A Special Case

- Each option o_i is associated with a random variable X_i.
- The random variables X_i take values in a set Ω that is only partially ordered by a preference relation $\preceq$.
- $y_{i,j} = \mathbf{P}(X_i \prec X_j) + \frac{1}{2}\Big(\mathbf{P}(X_i \sim X_j) + \mathbf{P}(X_i \perp X_j)\Big)$
- $\bar{y}_{i,j} = \frac{1}{n_i n_j}\sum_{\ell=1}^{n_i}\sum_{\ell'=1}^{n_j}\Big[\mathbb{I}\{x_i^{\ell} \prec x_j^{\ell'}\} + \frac{1}{2}\big[\mathbb{I}\{x_i^{\ell} \sim x_j^{\ell'}\} + \mathbb{I}\{x_i^{\ell} \perp x_j^{\ell'}\}\big]\Big]$



Theorem (Expected sample complexity for SE ranking)

Let $\mathcal{O} = \{o_1, \dots, o_K\}$ be a set of options. Assume $o_i \prec^{\mathrm{SE}} o_j$ iff $i < j$ without loss of generality and $y_i \neq y_j$ for all $1 \leq i \neq j \leq K$. Let
$$b_i = \left\lceil \left(\frac{4}{y_i - y_{K-\kappa+1}}\right)^2 \log\frac{2K^2 n_{\max}}{\delta}\right\rceil \text{ for } i \in [K-\kappa] \text{ and}$$
$$b_j = \left\lceil \left(\frac{4}{y_j - y_{K-\kappa}}\right)^2 \log\frac{2K^2 n_{\max}}{\delta}\right\rceil \text{ for } j = K-\kappa+1, \dots, K.$$
Then, whenever $n_{\max} \geq b_{K-\kappa} = b_{K-\kappa+1}$, $\mathrm{PBR\text{-}\mathbf{SE}}$ terminates after $\sum_{i \neq j} b_i = \sum_{i=1}^{K-\kappa}(K-1)b_i + \sum_{j=K-\kappa+1}^{K}(K-1)b_j$ pairwise comparisons and outputs the optimal solution with probability at least $(1-\delta)$.

References

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